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Fluid flow in an inhomogeneous granular medium☆

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ABSTRACT

The seepage of a compressible fluid in an inhomogeneous undeformable granular medium is investigated. It is assumed that the fluid flow in a porous space is described by the Navier–Stokes equations. It is shown that, in the case of an inhomogeneous velocity field, a tensor of additional effective stresses occurs in connection with the transfer of fluid particles in a transverse direction when flow occurs around the granules of the medium in a longitudinal direction. Using the fundamental propositions of Reynolds' averaging theory and Prandtl's mixing path, the structure of the effective viscosity coefficient is determined and hypotheses are formulated which enable it to be assumed to be independent of the flow velocity. It is established by comparison with experimental data that the effective viscosity coefficient can exceed the viscosity coefficient of the flowing fluid by an order of magnitude. The equations of average motion are obtained, which in the case of an incompressible fluid have the form of the Navier–Stokes equations with body forces proportional to the velocity. It is established that, in addition to the well-known dimensionless flow numbers, there is a new number which characterizes the ratio of the Darcy porous drag forces to the effective viscosity forces. The proposed equations are extended to the case of the flow of an aerated fluid. The components of the angular momentum vector are used as the required functions instead of the components of the velocity vector. This enables a solving system of equations to be obtained, which, apart from the notation, is identical with the similar equations for the case of an incompressible fluid. The solution of a new problem of the fluid flow in a plane channel with permeable walls is presented using three models: Darcy's law for an incompressible and aerated fluid, and also of an aerated fluid taking the effective viscosity into account. It is established that, for the same pressure drop, the maximum flow rate corresponds to Darcy's law. Compressibility leads to its reduction, but by simultaneously taking into account the compressibility and the effective viscosity one obtains minimum values of the flow rate. The effective viscosity and aeration of the fluid has a considerable effect on the flow parameters.

Numerous experimental papers[1–6] show that, for a fluid flow through a granular medium with variable porosity, when the flow field is inhomogeneous, the velocity profiles that occur differ considerably from the profiles predicted by Darcy's law. As a rule, experiments have been carried out in vertical circular tubes, containing a layer of granular material, peculated by a fluid with constant velocity distribution. At the layer exit considerable changes in the velocity profile were observed: the presence of a minimum in the centre of the tube, a maximum close to the wall, and zero velocity on the wall itself (i.e., so-called "ears" have been observed).

Characteristic experimental results[1] for a tube of radius $R = 3.81$ cm, filled with spherical particles of diameter $d = 0.64$ cm, are shown on the upper graph of Fig. 1 by the points, where υ_m is the fluid velocity, averaged over the cross section. On the same graph, the dashed curve represents an attempt to process these results using Darcy's law, taking into account the experimentally determined change in the porosity ε over the radius (the continuous curve in the lower graph of Fig. 1). The experimental results are in good agreement with Darcy's law in the central part of the tube but differ considerably in the boundary region.

This phenomenon has been mainly investigated in two directions. In the earlier investigations,[1,2] when processing the experimental results, Prandtl's formula was directly invoked for the turbulent shear stresses. Since, in Prandtl's theory, the shear stresses depend on the rate of deformation of the shear component according to a square law, the flow pattern must vary when the mean flow velocity changes. However, no such changes have been observed experimentally, and the authors have got round this contradiction by determining a missing length for each individual experiment.

Later[5,7] the inhomogeneity of the flow field was explained by deformation of the granular layer due to the action of the flowing fluid and buckling of the supporting net. A theory of the deformation of a granular medium was used in Ref. 7.

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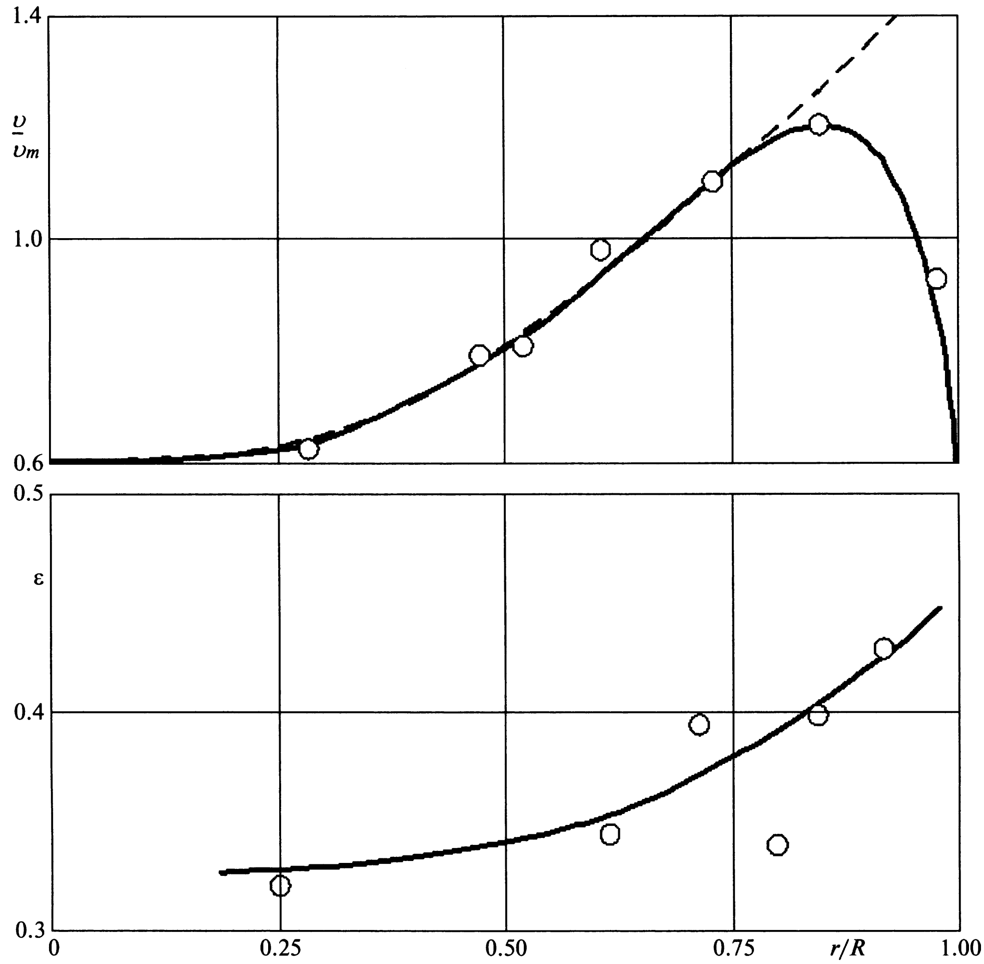

Fig. 1.

The view has been expressed in Refs. 4 and 5 that inhomogeneities of the velocity profile only arise on the free boundary of the layer. The correctness of Darcy's law for describing flows with a variable velocity profile was not in question, despite its obvious breakdown in the boundary region.

Without excluding the influence of the above factors on the fluid flow field, we will show that the experimentally observed results can be explained by introducing the idea of a transferred effective viscosity, similar in its physical nature to additional Reynolds stresses. Note that Eqs (1.10) then obtained when inertial terms are neglected is formally similar to the equations proposed by Brinkman[8,9] when investigating the flow of a solvent through tangled macromolecules consisting of long chains, considered as a porous medium. The usual derivation of Brinkman's equations consists of using Stokes' equations of a viscous fluid with external mass forces continuously distributed in the volume. The latter were assumed to be related to the velocity vector by Darcy's law. As Brenner has pointed out,[10] procedures of this kind are logically unjustified, since they combine equations describing two different continua. We will show in Section 1 that Eqs (1.10) are free from this drawback.

We note a recently published book,[11] in which, on the basis of fundamental concepts of continuum mechanics, fundamental propositions of the mechanics of fluids and multiphase media are described in detail. The elements of hydrostatics are discussed, and different forms of flow of ideal and viscous fluids are considered, as well as the fundamental ideas of the turbulence theory and the theory of similarity and dimensional analysis. A hydrodynamic theory of the seepage of fluids in uniform and nonuniform, isotropic and anisotropic media is given.

1. The transport phenomenon and the effective viscosity for flow in an inhomogeneous granular medium

When a compressible fluid flows through a porous medium, Darcy's law and the continuity equations are usually employed to determine the flow field. In an orthogonal Cartesian system of coordinates x_i ($i = 1, 2, 3 \to x, y, z$) these relations take the form

$$\mu\alpha\upsilon_i = -\frac{\partial P}{\partial x_i}, \quad \frac{\partial\rho}{\partial t} + \frac{\partial\rho\upsilon_i}{\partial x_i} = 0 \tag{1.1}$$

In Eqs (1.1) υ_i are the components of the mean-velocity vector in an elementary area, normal to the corresponding coordinate axis, μ is the coefficient of viscosity of the flowing fluid, ρ is its density, P is the pressure and α is the hydraulic drag, which is the inverse of the Darcy seepage factor. Equations (1.1) must be supplemented by the equation of state, which defines the relation between the density ρ and the pressure P. This case will be considered in Section 4. In the second formula of (1.1) and elsewhere summation is assumed over repeated indices.

Although Darcy's law was obtained experimentally and was treated as an independent law of nature, the first formula of (1.1) can be obtained if, following Zhukovskii,[12] we assume that the fluid, flowing in a porous space, is an ideal Euler fluid, on which fictitious porous drag mass forces $\mu\alpha\upsilon_i$ act. Euler's equations can then be written as[13]

$$\rho\frac{\partial\upsilon_i}{\partial t}+\rho\upsilon_j\frac{\partial\upsilon_i}{\partial x_j}+\mu\alpha\upsilon_i=-\frac{\partial P}{\partial x_i} \tag{1.2}$$

If we carry out the simplest averaging of Eqs (1.2) assuming that υ_i is the average velocity on elementary areas, normal to the coordinate axis considered, and inertial forces are neglected, we obtain Darcy's law (1.1).

A justification of Zhukovskii's hypothesis regarding the structure of the bulk porous drag force can be obtained using the methods and approaches developed by Slichter, on the assumption that the granular medium, through which the incompressible viscous percolates, consists of spherical particles of the same diameter, and that the centres of each of the eight adjoining spheres are at the vertices of a rhombohedron. These assumptions enable the shape of the porous channel, in which the elementary fluid particles move, to be determined. The actual porous channel obtained was replaced by a fictitious cylindrical channel of definite dimensions and cross section. Further, using the well-known analytical solution for laminar flow in such a channel, the average velocity component was found, i.e., the coefficient α in the first formula of (1.1) was determined.

An independent statement of the results was obtained by Slichter,[13] and there have also been numerous investigations by other researchers, devoted to a theoretical determination of the coefficient α. It follows from these that this coefficient can be represented in the form

$$\alpha = f(\varepsilon)/d^2 \tag{1.3}$$

where d is the characteristic size of a granule or pore and $f(\varepsilon)$ is a dimensionless function of the porosity ε, which depends on the shape of the granules and the type of packing.

In a porous medium, described by Darcy's law, the full no-slip conditions on the boundary surfaces may not be satisfied. Hence, neither Darcy's law itself nor its well-known extensions (the non-linear dependence of the pressure gradient on the velocity vector, the presence of turbulence and stagnation zones at points of contact of the particles, etc.) cannot explain the phenomena in the boundary region described above.

When fluid flows around the particles of a granular medium in a longitudinal direction, pulsating components of the velocity occur in the transverse direction, which are ignored by Darcy's law. If the average flow velocity changes in a transverse direction, these pulsations transfer additional momenta from layer to layer, which leads to the occurrence of effective viscosity forces. This phenomenon can be investigated if it is assumed that a viscous fluid flows in the porous space, the equation of motion of which[14] cab be represented in the form

$$\rho\frac{\partial\upsilon_i}{\partial t}+\rho\upsilon_j\frac{\partial\upsilon_i}{\partial x_j}+\frac{\partial\tau_{ij}}{\partial x_j}=-\frac{\partial P}{\partial x_i} \tag{1.4}$$

where τ_{ij} are the components of the viscous stress tensor.

We will write the components of the fluid flow velocity in the granular medium υ_i in the form of the sum of the average smoothed values $\bar{\upsilon}_i$ and pulsating components υ'_i, the average values of which are equal to zero, i.e., $\upsilon_i=\bar{\upsilon}_i+\upsilon'_i$. Substituting these values into Eq. (1.4) and carrying out well-known averaging operations,[14] we obtain the equations of averaged motion

$$\rho\frac{\partial\bar{\upsilon}_i}{\partial t}+\rho\bar{\upsilon}_j\frac{\partial\bar{\upsilon}_i}{\partial x_j}+\mu\alpha\bar{\upsilon}_i=-\frac{\partial\bar{P}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(-\rho\overline{\upsilon'_i\upsilon'_j}\right) \tag{1.5}$$

which differ from the initial equations (1.4) in the fact that, in accordance with Zhukovskii's hypothesis, the stresses due to the fluid viscosity were averaged over the bulk force of the Darcy porous drag, while a consideration of the inertial forces led to the occurrence of six terms

$$\tau_{ij}^R=\rho\overline{\upsilon'_i\upsilon'_j} \tag{1.6}$$

called additional stresses.

When deriving Eqs (1.5) the physical nature of the velocity pulsations that occur was not stipulated; the fact that they existed was only important. In the case considered, unlike turbulent flows, the pulsations are due to the presence of granules in the flow field, and hence it is necessary to establish whether flow modes exist for which the stresses, due to the effective viscosity, are comparable with the other forces acting on the fluid.

To find the relation between the stresses (1.6) and the average velocity field in the flow (the closure problem) we will use the fundamental propositions of Prandtl's mixing path theory. Consider a steady plane translational flow parallel to the x axis, the mean velocity of which $\upsilon_x=u(y)$ depends on the transverse coordinate y. The pulsating component of the longitudinal velocity u' is taken in the form $u'\approx l'\,du/dy$, where l' is Prandtl's mixing length. The pulsating velocity component $\upsilon'_y=\upsilon'$ transverse to the average flow is taken as the transfer velocity. Substituting these quantities into formula (1.6), we obtain

$$\tau_{xy}^R=-\rho\overline{(\upsilon'u')}=\rho\overline{(\upsilon'l')}du/dy$$

It can be seen that $\overline{(\upsilon' l')} = \text{const}$, if we assume that l' will be shorter the greater the pulsating velocity υ'. Assuming that this assertion holds for the whole flow field (Boussinesq's hypothesis), we will have

$$\tau_{xy}^{R} = \mu_e \frac{d\bar{u}}{dy}, \quad \mu_e = \rho\overline{(\upsilon' l')} \tag{1.7}$$

The quantity μ_e can be treated as an effective (transfer) viscosity coefficient, which occurs when the fluid flow through the porous medium is spatially inhomogeneous. The effective viscosity coefficient can conveniently be represented in the form $\mu_e = \mu\beta(\varepsilon)$, where $\beta(\varepsilon)$ is a dimensionless porosity function. In the general case of an isotropic medium, the effective stresses form a symmetrical second-rank tensor, whose components, based on Eqs (1.7), can be expressed in terms of the average velocities in the form

$$\tau_{ij}^{R} = \mu\beta(\varepsilon)\left(\frac{\partial\bar{\upsilon}_i}{\partial x_j} + \frac{\partial\bar{\upsilon}_j}{\partial x_i}\right) \tag{1.8}$$

while the components of the complete stress tensor will have the form

$$\tau_{ij} = \mu(1+\beta(\varepsilon))\left(\frac{\partial\bar{\upsilon}_i}{\partial x_j} + \frac{\partial\bar{\upsilon}_j}{\partial x_i}\right) \tag{1.9}$$

By comparing the theoretical results with experimental data it will be shown below that the effective viscosity coefficient exceeds the viscosity coefficient μ by a factor of a hundred. Taking this into account and substituting expression (1.9) into Eq. (1.5), we obtain for an incompressible fluid

$$\rho\left(\frac{\partial\upsilon_i}{\partial t} + \upsilon_j\frac{\partial\upsilon_i}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left[\mu\beta(\varepsilon)\left(\frac{\partial\upsilon_i}{\partial x_j} + \frac{\partial\upsilon_j}{\partial x_i}\right)\right] + \mu\alpha\upsilon_i = -\frac{\partial P}{\partial x_i}, \quad \frac{\partial\upsilon_j}{\partial x_j} = 0 \tag{1.10}$$

In formulae (1.10) and below we omit the bar over average quantities and assume that summation is carried out over repeated indices. Equations similar to (1.10) were obtained in a somewhat different form for the plane case in Ref. 15 from phenomenological considerations as extensions of Stokes' equations.

Equations similar to (1.10) were also obtained in Refs 16 and 17 on the assumption that the motion of the fluid is described by the Navier–Stokes equation. The important result that the occurrence of an effective (transfer) viscosity is practically independent of the natural viscosity of the fluid has not been previously mentioned.

Equations (1.10) were extended in Ref. 16 to anisotropic flows, which occur due to the natural or technogenic anisotropy of a porous space. The source of technogenic anisotropy may be ordered packing of the granular layer by granules of special shape (for example, Raschig rings), which enable the flow parameters to be directionally changed[18] and enable specified optimality criteria to be satisfied.

Equations (1.10) can be reduced to dimensionless form by choosing certain constant quantities T, h, V and P, characteristic for the flow, as scales of time, length, velocity and pressure. Denoting the corresponding dimensionless quantities by a prime and introducing, as is usually done, the dimensionless Strouhal, Euler and Reynolds similitude numbers

$$\text{Sh} = \frac{h}{VT}, \quad \text{Eu} = \frac{P}{\rho V^2}, \quad \text{Re} = \frac{hV\rho}{\mu\beta}$$

we can write Eqs (1.10) in the form

$$\text{Sh}\frac{\partial\upsilon_i'}{\partial t'} + \upsilon_j'\frac{\partial\upsilon_i'}{\partial x_i'} - \frac{1}{\text{Re}}\left[\frac{\partial}{\partial x_j'}\left(\frac{\partial\upsilon_i'}{\partial x_j'} + \frac{\partial\upsilon_j'}{\partial x_i'}\right) - \text{So}\,\upsilon_i'\right] = -\text{Eu}\frac{\partial P'}{\partial x_i'}, \quad \text{So} = \frac{\alpha}{\beta}h^2 = \frac{h^2 f(\varepsilon)}{d^2\beta(\varepsilon)} \tag{1.11}$$

The dimensionless number So represents the ratio of the Darcy porous drag forces to the effective viscosity forces. This was first proposed in a somewhat different form in Ref. 15. The effect of the So number on the flow pattern will be illustrated below using a number of examples.

2. Flow in plane and axisymmetric channels

We will consider, as an example, the fluid flow in a porous medium, situated in a long plane channel of height $2h$, assuming the porosity to be constant. It follows from this condition that the coefficients α and β are also constant. Directing the x axis along the middle line of the channel and substituting the values $\upsilon_y = \upsilon_z = 0$, $\upsilon_x = u(y)$ into Eq. (1.10), we obtain

$$\beta\frac{d^2u}{dy^2} - \alpha u = \frac{\theta}{\mu}, \quad \theta = \frac{dP}{dx} = \text{const}$$

The solution of this equation, which satisfies the boundary conditions $u = 0$ when $y = \pm h$, will be

$$u = \mu\frac{\theta}{\alpha}\left(1 - \frac{\text{ch}\chi\bar{y}}{\text{ch}\chi}\right), \quad \chi = \sqrt{\frac{\alpha}{\beta}}h = \sqrt{\frac{f(\varepsilon)}{\beta(\varepsilon)}}\frac{h}{d} = \sqrt{\text{So}}, \quad \bar{y} = \frac{y}{h} \tag{2.1}$$

Graphs of the dimensionless velocity $\tilde{u} = u(\bar{y})/u(0)$ for a series of increasing values of the number So are shown in Fig. 2, where the horizontal dashed line represents the solution in the case of Darcy's law. It follows from the graph that the solution becomes Darcy's

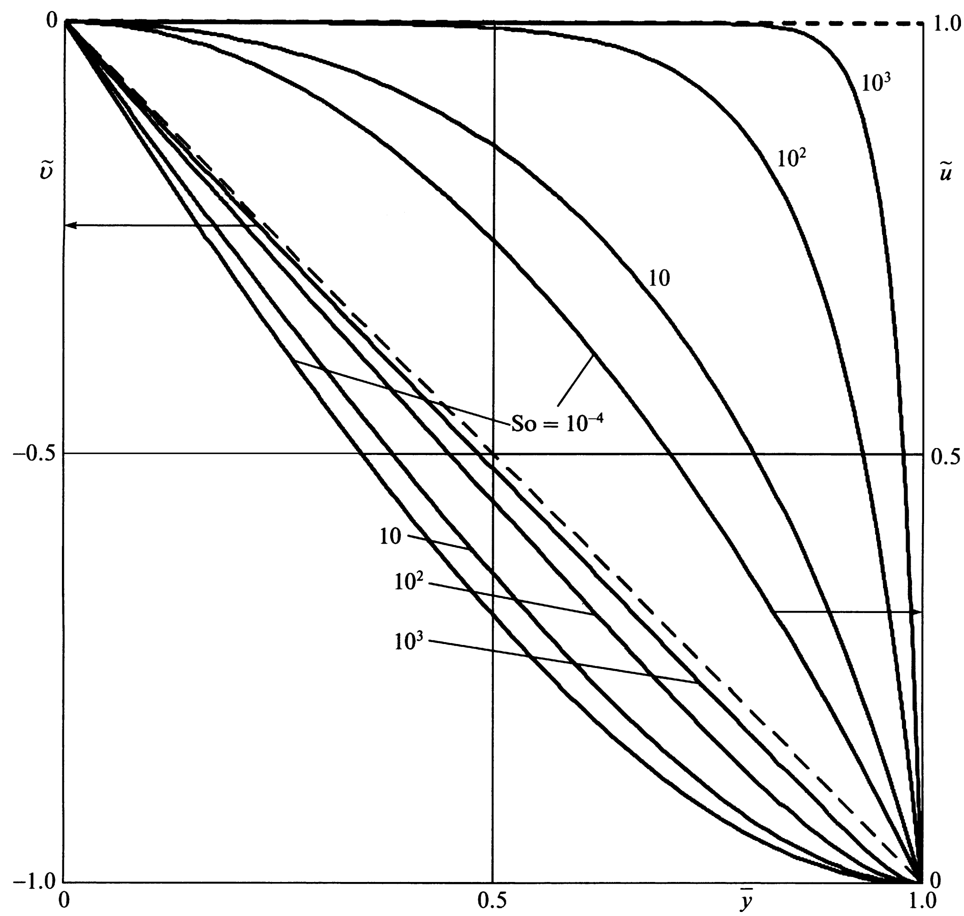

Fig. 2.

solution $u_0 = \theta/\mu\alpha$) when the So number increases without limit, i.e., when, for a fixed porosity ε, the size of the granules is much less than the channel width. Formulae (2.1) show that when $So \to 0(\alpha \to 0)$, the solution becomes a Poiseuille parabola.

Knowing the velocity distribution along the channel height y, we obtain the fluid flow rate in an arbitrary cross section x in the form

$$Q = \frac{2h^3(1-\lambda)}{\mu\bar{\alpha}}\theta, \quad \bar{\alpha} = \alpha h^2, \quad \lambda = \frac{\text{th}\chi}{\chi} = \frac{\text{th}\sqrt{\text{So}}}{\sqrt{\text{So}}} \tag{2.2}$$

It follows from formulae (2.2) that, for a specified pressure drop, the flow depends considerably on the dimensionless So number and decreases from the maximum value $Q_0 = 2h^3\theta/(\mu\bar{\alpha})$ corresponding to Darcy's law, as the So number decreases by virtue of the increase in the influence of the effective viscosity. This result has an important practical value since it enables one to judge, from the experimentally determined dependence of the flow rate on the pressure drop, the actual rheological properties of the flowing fluid.

A graph of the dimensionless flow rate $\bar{Q} = Q_\mu/(h^3\theta) = 2(1-\lambda)/\bar{\alpha}$ as a function of the dimensionless parameters β and $\bar{\alpha}$ is shown in Fig. 3. The curve in the $\beta = 0$ plane corresponds to Darcy's law $\bar{Q} = 2/\bar{\alpha}$, while in the plane $\bar{\alpha} = 0$ it corresponds to the Navier–Stokes equations $\bar{Q} = 2/(3\beta)$ with a viscosity coefficient of $\mu\beta$.

To determine how the effective viscosity coefficient depends on the porosity,[17] the above-mentioned experimental results[1] for velocity profiles in a tube of radius R, filled with a granular medium with a characteristic particle size d, were used.

Equation (1.10) has the following form in the axisymmetric case (the x coordinate is directed along the tube axis)

$$\beta\frac{d^2\upsilon}{d^2r} + \left(\frac{\beta}{r} + \frac{d\beta}{dr}\right)\frac{d\upsilon}{dr} - \alpha\upsilon = \frac{1}{\mu}\frac{\partial P}{\partial x} \tag{2.3}$$

The coefficient α (1.3) is given by the Kozeny formula, in which the experimental dependence of the porosity on the radius is used (the lower graph in Fig. 1). Starting from the mechanism by which the effective velocity occurs, the following form of the relation $\beta = \beta(\varepsilon)$ has been proposed

$$\beta(\varepsilon) = A/\varepsilon^\lambda \tag{2.4}$$

where A and λ are dimensionless coefficients, which are independent of the porosity of the medium and the size of the granules.

Equation (2.3) was solved numerically for several values of the tube diameter and the size of the packing granules. It was found that the value of the parameter A varies in the range from 50 to 60, while the value of the parameter λ varies in the range from 3.5 to 4. On the upper graph in Fig. 1 the points represent experimental data for a tube of radius $R = 3.81$ cm, filled with particles of diameter $d = 0.64$ cm,

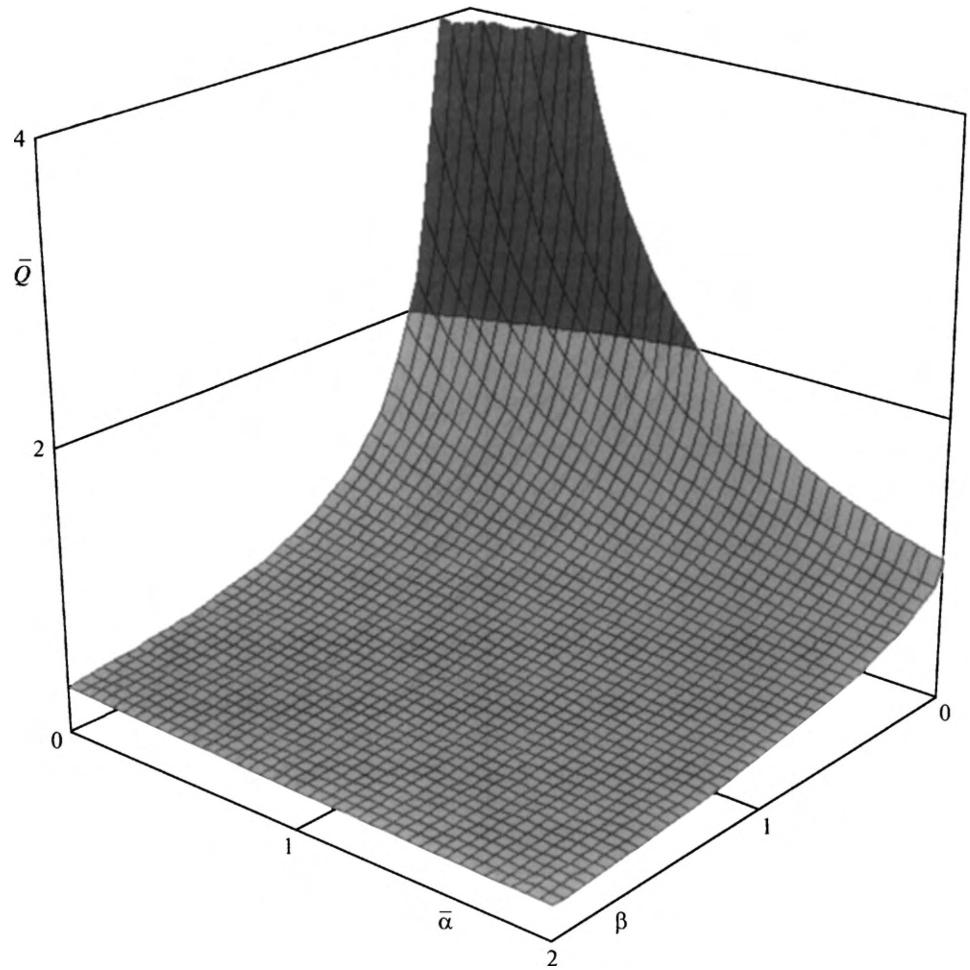

Fig. 3.

while the continuous curve represents the velocity profile obtained. The proposed relation (2.4) describes the experimental data quite well over a wide range of the parameters and shows that, for the usual values of the porosity of 0.3–0.45, the value of β is of the order of 10^2–10^3, i.e., the value of the effective viscosity is many times greater than the value of the natural viscosity of the fluid, so the latter can be neglected.

As was noted above, a number of researchers have shown that non-uniformity of the velocity field only occurs in the outer surface layer of the granules. To check this assertion, we considered the problem of the flow in a long channel of circular cross section in which there was a layer of granular material.[19] The change in the porosity along the radius was also taken in the form shown on the lower graph of Fig. 1. We used the Navier–Stokes equations in the parts of the channel that were free from granules. The numerical solution of the problem showed that perturbations of the velocity profile occur before the fluid enters the layer. They reach maximum values inside the layer and decrease somewhat in the exit section.

3. Flow in channels with permeable walls

The fluid flow in channels with permeable walls is encountered in many technological processes and is of particular interest for problems involving the extraction of hydrocarbon raw material, where the high-yield method has become widely used, involving drilling horizontal boreholes with subsequent use of the hydrofracturing procedure.

We will consider a plane channel of constant width $2h$ and length a. We will direct the x axis along the middle line of the channel so that the points $x = 0$ and $x = a$ correspond to the beginning and end of the channel sections. At the boundaries of the channel, with $y = \pm h$, we will specify a constant velocity w, which is independent of the x coordinate: $\upsilon(x, \pm h) = \mp w = const$. We will denote the pressure at $x = a$ by P_w, and at $x = 0$ by $P_r (P_w < P_r)$.

Assuming, in Eqs (1.10), that the forces of inertia of the average motion are negligibly small compared with the porous drag and effective viscosity forces, we obtain

$$\beta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\alpha u=\frac{1}{\mu}\frac{\partial P}{\partial x},\quad \beta\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}\right)-\alpha \upsilon=\frac{1}{\mu}\frac{\partial P}{\partial y},\quad \frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=0 \tag{3.1}$$

where u and υ are the components of the velocity vector along the x and y axes respectively.

We will consider the solution of the problem using Darcy's law. Assuming $\beta = 0$ in Eqs (3.1) and integrating the system of equations obtained, we will have

$$u = wx/h, \quad \upsilon = -wy/h, \quad P = P_r - w\bar{\alpha}\mu\left(x^2 - y^2\right)/\left(2h^3\right) \tag{3.2}$$

Hence it follows that the fluid flow rate in the section $x = a$ is related to the pressure drop $\Delta P = P_r - P_W$ by the equality $Q_W = 4h^3\Delta P/(a\bar{\alpha}\mu)$. Note that this quantity, for the same pressure drop, is double the flow rate Q_0 corresponding to the flow in a channel with impermeable walls.

Returning to the complete equations (3.1), we can verify that the velocity components u and υ, and the pressure P, which satisfy these equations and the boundary conditions, have the form

$$u = \frac{wx}{h(1-\lambda)}\left(1 - \frac{\operatorname{ch}\chi\bar{y}}{\operatorname{ch}\chi}\right), \quad \upsilon = \frac{w}{1-\lambda}\left(\lambda\frac{\operatorname{sh}\chi\bar{y}}{\operatorname{sh}\chi} - \bar{y}\right), \quad P = P_r - \frac{w\bar{\alpha}\mu}{2h^3(1-\lambda)}\left(x^2 - y^2\right) \tag{3.3}$$

where we have used the notation (2.1) and (2.2).

The first formula of (3.3) shows that the form of the relation $\tilde{u}(\bar{y}) = u(x, \bar{y})/u(x, 0)$ is the same as in the case of a channel with impermeable walls, while the graph for $\tilde{u}(\bar{y})$ is identical with the graph for $\tilde{u} = \tilde{u}(\bar{y})$, shown in Fig. 2. The relation $\tilde{\upsilon}(\bar{y}) = \upsilon(\bar{y})/w$ is shown in Fig. 2 for a series of values of the So number. These graphs become graphs corresponding to Darcy's law as So → 0 (the dashed curves in Fig. 2: for $\tilde{u} = \tilde{u}(\bar{y})$ the horizontal line and for $\tilde{\upsilon} = \tilde{\upsilon}(\bar{y})$ the diagonal line), and as So → 0 they correspond to the Navier–Stokes equations with viscosity coefficient $\beta\mu$.

It follows from the last formula of (3.3) that the total fluid flow rate in the channel $Q_W = Q(a) = 2wa$ is related to the pressure drop $\Delta P = P_r - P_W$ and, in dimensionless form, the relation

$$\bar{Q} = Q_w\mu a/\left(2h^3\Delta P\right) = 2(1-\lambda)/\bar{\alpha}$$

is identical with the relation $\bar{Q}(\bar{\alpha}, \beta)$, shown in Fig. 3.

4. The flow of an aerated liquid

Consider the steady flow of an aerated liquid in an undeformed granular medium. We will introduce the momentum vector $\mathbf{k} = \rho\mathbf{V}$ with components $k_i = \rho\upsilon_i$. We will assume that the flow in the porous space is described by Eqs (1.4) and the continuity equation, the second formula of (1.1). In this case they take the form

$$k_j\frac{\partial\upsilon_i}{\partial x_j} + \rho\frac{\partial\tau_{ij}}{\partial x_j} = -\rho\frac{\partial P}{\partial x_i}, \quad \frac{\partial k_j}{\partial x_j} = 0 \tag{4.1}$$

Using the new function q, such that

$$dq = \rho dP \tag{4.2}$$

and taking into account the last equation of (4.1),we obtain

$$\frac{\partial k_j\upsilon_i}{\partial x_j} + \rho\frac{\partial\tau_{ij}}{\partial x_j} = -\frac{\partial q}{\partial x_i} \tag{4.3}$$

From the same considerations as in Section 1 when analysing the velocity field, we can represent the components of the momentum vector k_i in the form

$$k_i = \bar{k}_i + k_i' \tag{4.4}$$

where $\bar{k}_i$ are the average values of k_i, while k_i' is their pulsating components. Substituting (4.4) into Eqs (4.3) and into the last equation of (4.1), we obtain after averaging

$$\bar{k}_j\frac{\partial\bar{k}_i}{\partial x_j} + \frac{\partial\overline{k_i'k_j'}}{\partial x_j} - \mu\alpha\bar{k}_i = \frac{\partial\bar{q}}{\partial x_i}, \quad \frac{\partial\bar{k}_j}{\partial x_j} = 0 \tag{4.5}$$

In deriving the averaged equations (4.5) we took into account the fact that, as a result of the averaging of Eqs (4.3), terms containing viscous stresses due to the Zhukovskii hypothesis,[12] are converted into the mass Darcy porous drag force. We have also assumed that the compressibility of the fluid can be neglected on the left-hand sides of Eqs (4.3), and we have only taken it into account in the averaged continuity equation. This approach is widely used[13] in the theory of the compressible fluid flow in porous media.

The four equations (4.5) contain three components of the momentum vector $\bar{k}_i$ and the quantity $\bar{q}$ as the desired functions. However, this system of equations is not closed, since it contains the terms $\overline{k_i'k_j'}$. To establish their dependence on the average values $\bar{k}_i$ we will use, as previously, the Prandtl mixing path theory. As in Section 1, for this purpose we will consider a plane translational flow, parallel to the x

axis, the average momentum of which $k_x = k_x(y)$ depends on the transverse coordinate y. Repeating the discussion of Section 1, we obtain for the quantity $\psi_{ij} = \overline{k_i'k_j'}$, for $i = x$ and $j = y$,

$$\psi_{xy} = \overline{k_y'l'}\frac{d\bar{k}_x}{dy} = A\frac{d\bar{k}_x}{dy} \tag{4.6}$$

The coefficient of proportionality A can be taken to be a constant quantity if we assume that the pulsations l' will be less the greater the pulsations k_y'. Assuming this assumption to hold for all flow fields, equality (4.6) can be extended to the case of any three-dimensional motion and represented in the form

$$\psi_{ij} = \mu\eta\left(\frac{\partial\bar{k}_i}{\partial x_j} + \frac{\partial\bar{k}_j}{\partial x_i}\right), \quad \eta = \frac{A}{\mu} \tag{4.7}$$

Introducing these quantities into equations of motion (4.5) we obtain

$$k_j\frac{\partial k_i}{\partial x_j} + \alpha\mu k_i - \frac{\partial}{\partial x_j}\left[\mu\eta\left(\frac{\partial k_i}{\partial x_j} + \frac{\partial k_j}{\partial x_i}\right)\right] = -\frac{\partial q}{\partial x_i} \tag{4.8}$$

Here and below the bar above average quantities will be omitted. Equations (4.8), like Eqs (1.10), describe the change in the average momentum for the fluid flow through a granular medium. However, the required functions in Eqs (1.10) are the components of the velocity vector, whereas in Eqs (4.8) they are the components of the momentum vector.

The system consisting of the last equation of (4.5) and the three equations of (4.8) contain four required functions and enable us to determine the velocity field and the pressures of the compressible fluid, using Eq. (4.2), if we know how the density ρ depends on the pressure P.

Assuming the flow to be isothermal, this relation takes the form[13]

$$\bar{\rho} = \frac{\bar{P}}{\theta + (1-\theta)\bar{P}}, \quad \bar{\rho} = \frac{\rho}{\rho_0}, \quad \bar{P} = \frac{P}{P_0} \tag{4.9}$$

where P_0 and ρ_0 are the pressure and density for which the fluid phase (the liquid with the gas dissolved in it) and the gaseous phase (a gas in the form of fine dust particles, freely moving in the pores) is in an equilibrium state, and θ is an experimentally determined parameter. When $P < P_0$, by Henry's law, the gas separates out from the liquid phase, and the overall density per unit volume of the liquid phase is reduced, i.e., when $P < P_0$ we have $\rho < \rho_0$.

Substituting expressions (4.9) into Eq. (4.2), we obtain

$$\bar{q} = \frac{q}{\rho_0 P_0} = b\bar{P} - \theta b^2 \ln(\bar{P} + \theta b) + C, \quad b = \frac{1}{1-\theta} \tag{4.10}$$

where C is an integration constant.

As an example of the use of the equations obtained, we will again consider the solution of the above problem on the fluid flow in a plane channel. Unlike the discussion in Section 3, on the boundary of the channel $y = \pm h$ we will specify a constant mass flow rate ω, independent of the x coordinate, i.e., $k_y(\pm h) = \mp\omega = \text{const}$, and the value of the pressure function (4.2) at the points $x = 0$, $y = 0$ and $x = a$, $y = 0$ will be denoted by q_r and q_w respectively. Henceforth we will assume

$$P_0 = P_r, \quad \rho_0 = \rho_r, \quad q_0 = q_r$$

Then expression (4.10) then takes the form

$$\bar{q} = \frac{q_r}{P_r\rho_r} + b(\bar{P} - 1) - \theta b^2 \ln\frac{\bar{P} + \theta b}{1 + \theta b} \tag{4.11}$$

If the flow field is such that the average inertial forces and effective viscosity forces are much less than the porous drag forces, the first two terms in the first equation of (4.5) can be neglected, and we can obtain, together with the continuity equation of (the second formula of (4.5)) a system of four equations in the four required functions k_i and q. This system describes the flow of a compressible fluid in a granular medium in the Darcy law approximation, and its solution reduces to solving Laplace's equation for the function q.[13] In this case the solution of the above problem on the flow in a channel with permeable walls takes the form[20]

$$k_x = \omega x/h, \quad k_y = -\omega y/h, \quad q = q_r - \bar{\omega}(x^2 - y^2), \quad \bar{\omega} = \mu\bar{\alpha}\omega/(2h^3) \tag{4.12}$$

Formulae (4.11) and (4.12) show that the mass flow rate through the channel is given by the relation

$$Q_\rho = 2\omega a = 4h^3\Delta q/(\mu\bar{\alpha}a); \quad \Delta q = q_r - q_w \tag{4.13}$$

It follows from formula (4.11) that $\Delta q < \rho_r \Delta P$ always, and consequently for a specified pressure drop, the mass flow rate (4.13) obtained will always be less than the mass flow rate

$$Q_D = 4h^3\Delta P\rho_r/(\mu\bar{\alpha}a) \tag{4.14}$$

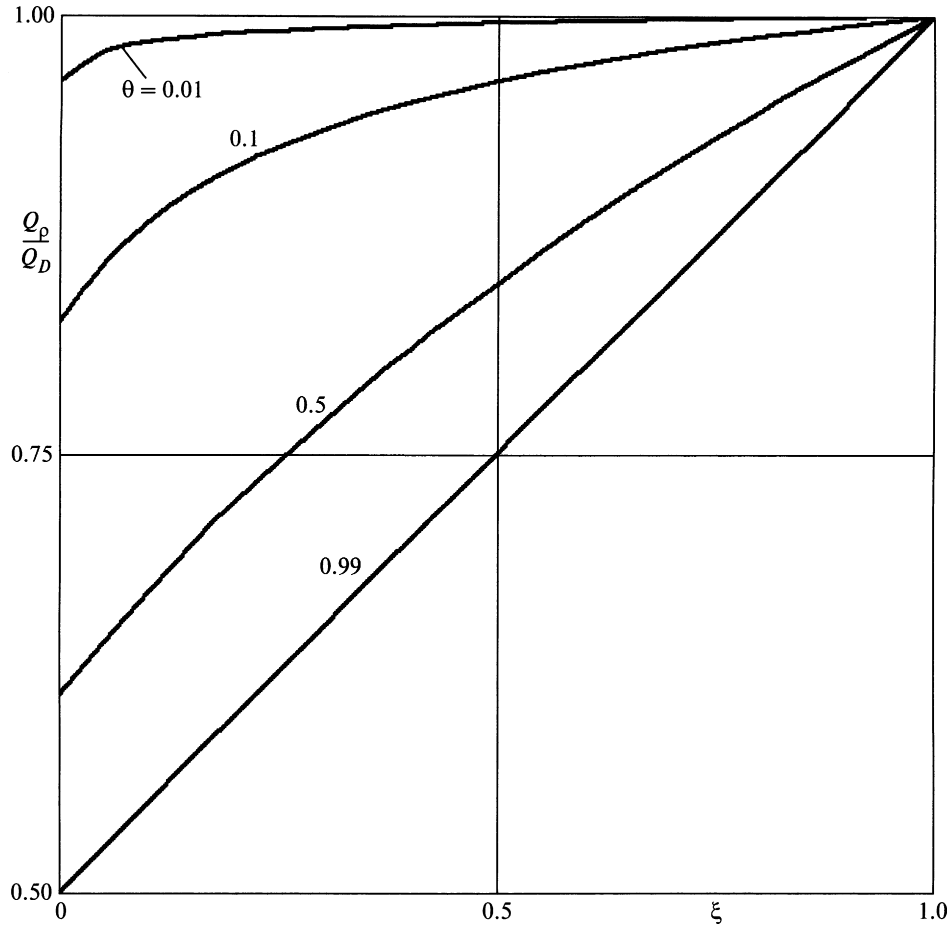

Fig. 4.

corresponding to Darcy's law for a fluid with a constant density ρ_r. The dependence of the ratio Q_ρ/Q_d on the value of the pressure-difference parameter $\xi = P_w/P_r$ is shown in Fig. 4 for a series of values of θ, where, when θ = 0 (an incompressible fluid), we have $Q_\rho = Q_D$. Note that, in dimensionless form, the relation

$$\bar{Q} = Q_\rho \mu a\Big/\left(2h^3\Delta q\right) = 2/\bar{\alpha}$$

coincides with the curve in the θ = 0 plane (Fig. 3).Fig. 4

Formula (4.11) and the last of formulae (4.12) enable us to obtain an equation which defines the pressure distribution $\bar{P} = \bar{P}(x, y)$, $\bar{P} = P/P_r$ in the form

$$b\left(1-\bar{P}\right) + \theta b^2 \ln\frac{\bar{P}+\theta b}{1+\theta b} = \tilde{x}^2 - \tilde{y}^2, \quad \frac{\tilde{x}}{x} = \frac{\tilde{y}}{y} = \sqrt{\frac{\bar{\omega}}{2h^3\rho_r P_r}} \tag{4.15}$$

In the space $\bar{P}, \tilde{x}, \tilde{y}$ the first formula of (4.15) defines a family of surfaces as a function of the parameter θ. Knowing the value of the pressure P_w in the exit section of the channel we can find the coordinate $\tilde{x}_w$ corresponding to it and obtain the relation $\bar{P} = \bar{P}(x/a)$, since $x/a = \tilde{x}/\tilde{x}_w$. A series of curves $\bar{P} = \bar{P}(\tilde{x}, 0)$ for a number of values of the parameter θ is shown in the upper part of Fig. 5. For example, for θ = 0.5 and $P_w = 0.2\,P_r$ it turned out that $\tilde{x}_w = 0.76$. A series of curves $\bar{\rho} = \bar{\rho}(\tilde{x}, 0)$ for the same values of θ are given in the lower part of Fig. 5, and we obtained a value $\bar{\rho}_w = \rho(\tilde{x}_w)/\rho_r = 0.34$ for the dimensionless density in the exit section of the channel. The corresponding points in both parts of Fig. 5 are shown by the light circles.

Bearing in mind that $y/a = \tilde{y}/\tilde{x}_w$, and specifying the ratio h/a, for each section x we can construct the relation $P = P(\tilde{y})$, which has a minimum when $\tilde{y} = 0$, i.e. on the axial line of the channel. Knowing the pressure field $\bar{P} = \bar{P}(\tilde{x}, \tilde{y})$, from formula (4.9) we can obtain the density distribution $\rho = \rho(x, y)$. In all the sections x the density has a minimum at the centre of the channel cross section and a maximum on the walls. Consequently, the concentration of the gaseous phase is a maximum along the axial line of the channel and a minimum on its walls. However, these differences are slight.

The graphs in Fig. 5 show that changes in the pressure and density along the channel axis depend very much on the compressibility parameter θ. The pressure and density are practically constant in the transverse direction.

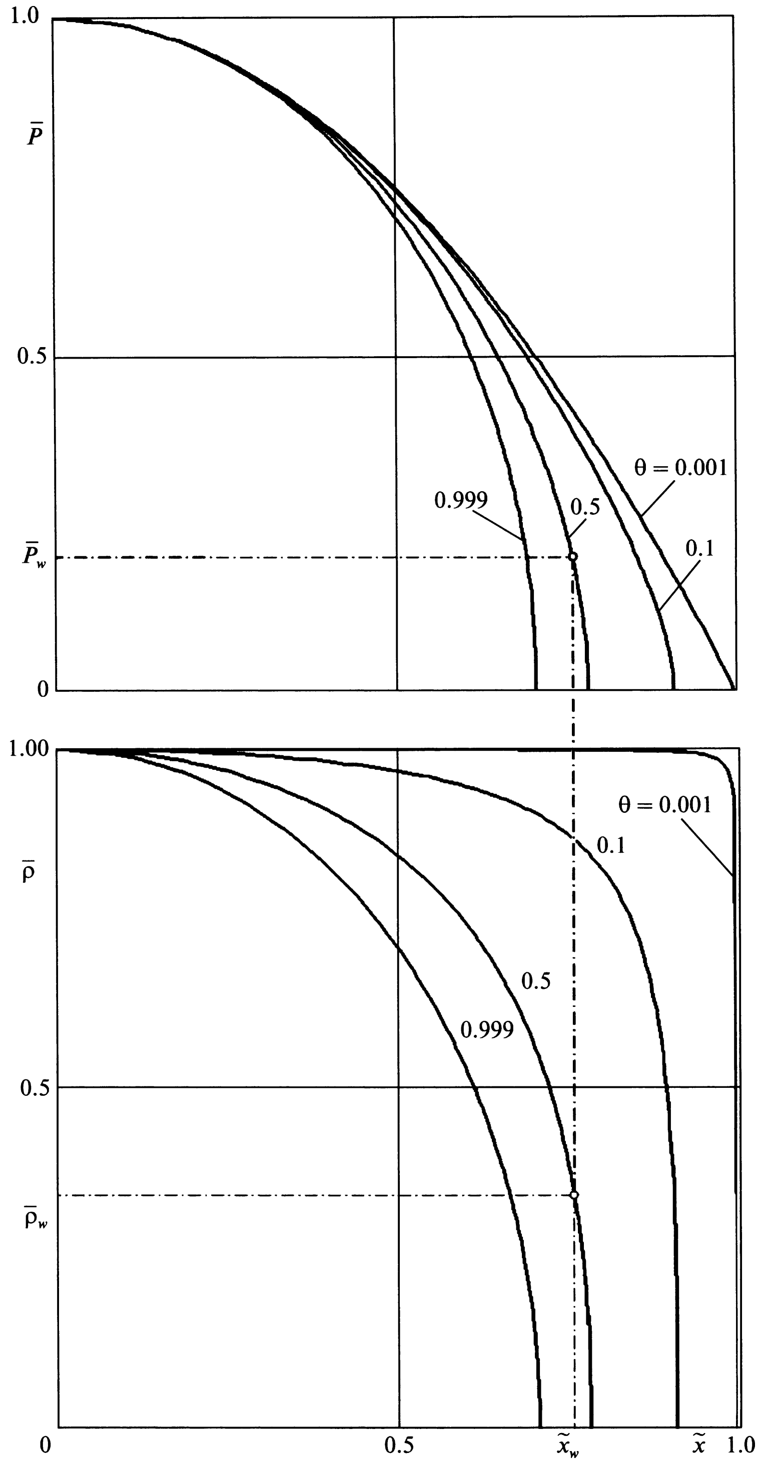

Fig. 5.

If, in Eqs (4.8), the term in the square brackets is retained and the averaged inertia forces are neglected, the corresponding system of equations for the plane case takes the form

$$\eta\left(\frac{\partial^2 k_x}{\partial x^2}+\frac{\partial^2 k_x}{\partial y^2}\right)-\alpha k_x=\frac{1}{\mu}\frac{\partial q}{\partial x},\quad \eta\left(\frac{\partial^2 k_y}{\partial x^2}+\frac{\partial^2 k_y}{\partial y^2}\right)-\alpha k_y=\frac{1}{\mu}\frac{\partial q}{\partial y},\quad \frac{\partial k_x}{\partial x}+\frac{\partial k_y}{\partial y}=0 \tag{4.16}$$

Equations (4.16) and the boundary conditions are identical, apart from the notation, with Eqs (3.1) and the boundary conditions considered in Section 3, and consequently, their solution, by analogy with formulae (3.3), can be written as

$$k_x = \frac{\omega x}{h(1-\lambda)}\left(1 - \frac{\operatorname{ch}\chi\bar{y}}{\operatorname{ch}\chi}\right), \quad k_y = \frac{\omega}{1-\lambda}\left(\lambda\frac{\operatorname{sh}\chi\bar{y}}{\operatorname{sh}\chi} - \bar{y}\right), \quad q = q_r - \frac{\omega\bar{\alpha}\mu}{2h^3(1-\lambda)}\left(x^2 - y^2\right) \tag{4.17}$$

where we have introduced the notation

$$\chi = \sqrt{\mathrm{So}} = \sqrt{\frac{\bar{\alpha}}{\eta}}, \quad \lambda = \frac{\operatorname{th}\chi}{\chi}$$

If we assume that the variables $\tilde{x}, \tilde{y}$ in (4.15) have the form

$$\frac{\tilde{x}}{x} = \frac{\tilde{y}}{y} = \sqrt{\frac{\bar{\omega}}{2h^3\rho_r P_r(1-\lambda)}}$$

the method of constructing the velocity and pressure field for the case considered is identical with the method described above for the case of Darcy's law. The difference between the graphs shown in Fig. 5 and the graphs for the case considered reduces solely to a change of scale along the corresponding axes.

Formulae (4.17) show that

$$\tilde{k}_x = k_x(x,\bar{y})/k_x(x,0) = \tilde{u}, \quad \tilde{k}_y = k_y(\bar{y})/\omega = \tilde{\upsilon}$$

and the graphs of $\tilde{k}_x$ and $\tilde{k}_y$ against the dimensionless coordinate $\bar{y}$ coincide with the graphs of $\tilde{u}$ and $\tilde{\upsilon}$, shown in Fig. 2, respectively.

It follows from the last formula of (4.17) that the overall mass flow rate of the fluid through the channel Q_η differs from the similar flow rate ignoring the effective viscosity (4.13) by the factor $(1-\lambda)$. Consequently, the graph of the ratio $Q_\eta/((1-\lambda)Q_D)$ against ξ coincides with the one shown in Fig. 4.

The value of λ varies from 0 to 1; $\lambda \to 0$ as $\chi \to 0$ (the condition $\alpha \to \infty$ corresponds to Darcy's law), and $\lambda - 1$ as $\chi \to 0$ (the condition $\alpha = 0$ corresponds to the absence of porous drag). For a specified value of the pressure difference $\Delta P = P_r - P_W$ the mass flow rate Q_ρ (4.13), corresponding to a compressible fluid, turns out to be less than the flow rate Q_D (4.14), corresponding to Darcy's law for an incompressible fluid, while the flow rate $Q_\eta = (1-\lambda)Q_\rho$ when both the compressibility and the effective viscosity are taken into account simultaneously turns out to be less than the flow rate Q_ρ, i.e., $Q_\eta < Q_\rho < Q_D$. Consequently, if both the compressibility and the effective viscosity are taken into account, there is a considerable increase in the overall hydraulic drag of the channel, which is important in practice.

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